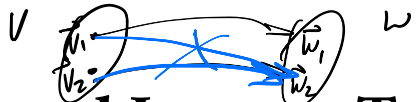

Due Fri

8.2 – Compositions and Inverse Transformations

Definition: If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W.
OR $T : V \to W$ is one-to-one if and only if for each vector $\mathbf{w}$ in the range of T, there is exactly one vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$.
OR $T : V \to W$ is one-to-one if and only if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

Definition: If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **onto** (or **onto W**) if every vector in W is the image of at least one vector in V.

Examples:

One-to-one and onto:

- Rotation operators on R^2 $(0 \le \theta < 2\pi)$
- $T: P_3 \to R^4$, where $T(a+bx+cx^2+dx^3) = (a,b,c,d)$
- $T: M_{22} \to R^4$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a,b,c,d)$

One-to-one but not onto:

- $T: P_n \to P_{n+1}$, where $T(\vec{p}) = T(p(x)) = xp(x)$

Not one-to-one:

- $D: C^1(-\infty,\infty) \to F(-\infty,\infty)$, where

  $D(\vec{f}) = f'(x)$ Ex: $D(x^2) = D(x^2+1) = 2x$

Theorem 8.2.1 If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

pf: (⇒) By Thm 8.1.1, $T(\vec{0}) = \vec{0}$ because T is linear. If T is one-to-one, then if $T(\vec{v}) = \vec{0}$, then $\vec{v} = \vec{0}$. Thus $\ker(T) = \{\vec{0}\}$.

(⇐) Let $\vec{u}, \vec{v} \in V$ such that $\vec{u} \neq \vec{v}$.

Then $\vec{u} - \vec{v} \neq \vec{0} \Rightarrow T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v}) \neq \vec{0}$

$\Rightarrow T(\vec{u}) \neq T(\vec{v}) \Rightarrow$ T is one-to-one ✓

#4 Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

a. $T : R^3 \to R^3$, where $T(x, y) = (x - y, y - x, 2x - 2y)$

b. $T : R^2 \to R^2$, where $T(x, y) = (0, 2x + 3y)$

c. $T : R^2 \to R^2$, where $T(x, y) = (x + y, x - y)$

a. $(x, y) \in \ker(T)$ if $x - y = 0$, $y - x = 0$, $2x - 2y = 0$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow y = x.$$

Not one-to-one.

b. $\ker(T) = \{(x, y) \mid 2x + 3y = 0\} \Rightarrow y = -\frac{2}{3}x.$

Not one-to-one.

C. If $\begin{matrix} x+y=0 \\ x-y=0 \end{matrix}$, then $2x=0 \Rightarrow x=0$
$\Rightarrow y=0$

$\ker(T)=\{(0,0)\}$. T is one-to one.

Theorem 8.2.X If $T : V \to W$ is a one-to-one linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent subset of V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is linearly independent.

<u>pf</u>: Suppose $c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_k T(\vec{v}_k) = \vec{0}$

Then because T is linear,

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = \vec{0}$$

Since T is one-to-one, $\sum_{i=1}^{k} c_i\vec{v}_i = \vec{0}$

since $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent,

$c_i = 0$, $0 \le i \le k$.

Thus, $\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_k)\}$ is linearly independant.

(Note: nullity(A) = dim $[\ker(T_A)]$)

#5 Determine whether multiplication by A is one-to-one by computing the nullity of A and then applying Theorem 8.2.1

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

(Note: for a one-to-one transformation T_A,
nullity$(A) = 0$).

a) $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

rank$(A) = 1 \Rightarrow$ nullity$(A) = 1$ $\boxed{\text{Not one-to-one}}$

b) rank ≤ 3
$\Rightarrow$ nullity ≥ 1
$\boxed{\text{not one-to-one}}$

Theorem 8.2.2 If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent.

a) T is one-to-one.
b) $\ker(T) = \{\mathbf{0}\}$.
c) T is onto [i.e., $R(T) = W$].

Theorem 8.2.3 If $T_A : R^n \to R^m$ is a matrix transformation, then
a) T_A is one-to-one if and only if the columns of A are linearly independent.
b) T_A is onto if and only if the columns of A span R^m.

#14 Use Theorem 8.2.3 to determine whether multiplication by A is one-to-one, onto, both, or neither. Justify your answer.

a. $A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$ d. $A = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

a) 2 cols can't span $R^3 \Rightarrow$ not onto

$c_1 \neq k\,c_2$ for any $k \Rightarrow$ independent columns $\Rightarrow$ one-to-one transf.

d) $\det(A) = 2 \neq 0 \Rightarrow$ one-to-one & onto

Theorem 8.2.4 Equivalent Statements (extends Theorems 4.9.8, 5.1.5 and 6.4.5, the last two of which we have not seen)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
g) $\det(A) \neq 0$.
h) The column vectors of A are distinct and linearly independent.
i) The row vectors of A are distinct and linearly independent.

j) The column vectors of A span R^n.
k) The row vectors of A span R^n.
l) The column vectors of A form a basis for R^n.
m) The row vectors of A form a basis for R^n.
n) A has rank n.
o) A has nullity 0.
p) The orthogonal complement of the null space of A is R^n.
q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$. ch 5.
r) $\lambda = 0$ is not an eigenvalue of A. [This is from Theorem 5.1.5.]
s) $A^T A$ is invertible. [This is from Theorem 6.4.5.]
t) The kernel of T_A is $\{\mathbf{0}\}$.
u) The range of T_A is R^n.
v) T_A is one-to-one.

new

Definition: If $T : V \to W$ is a one-to-one linear transformation with range $R(T)$, and if $\mathbf{w}$ is any vector in $R(T)$, then there is exactly one vector $\mathbf{v}$ in V for which $T(\mathbf{v}) = \mathbf{w}$. The **inverse of T** maps $\mathbf{w}$ back into $\mathbf{v}$ and is denoted by T^{-1}. (This is analogous to **inverse** in Section 1.9.)

Definition: (analogous to **composition** in Section 1.9)
If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the **composition** of T_2 with T_1, denoted by $T_2 \circ T_1$ (which is read "T_2 circle T_1") is the function defined by the formula $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$ where $\mathbf{u}$ is a vector in U.

Note that order matters when composing

Theorem 8.2.5 If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.

Theorem 8.2.6 If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then:
a) $T_2 \circ T_1$ is one-to-one.
b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

good proofs to practice

#38 Let $D(\mathbf{f}) = f'(x)$ and $J(\mathbf{f}) = \int_0^x f(t)dt$ be the derivative and integral linear transformations. Find $(J \circ D)(\mathbf{f})$ for

a. $\mathbf{f}(x) = x^2 + 3x + 2$

b. $\mathbf{f}(x) = \sin x$

a) $(J \circ D)\vec{f}) = J(D(f))$

$= J(2x+3) = \int_0^x (2t+3)dt$

$= x^2 + 3x$

Key concepts from 8.3:

Definition 1: A linear transformation $T : V \to W$ that is both one-to-one and onto is said to be an **isomorphism**, and W is said to be **isomorphic** to V.

Theorem 8.3.1 Every real n-dimensional vector space is isomorphic to R^n. [In this sense, every n-dimensional vector space is algebraically equivalent to R^n.]

Theorem 8.3.2 If S is an ordered basis for a vector space V, then the coordinate map $\mathbf{u} \xrightarrow{T} (\mathbf{u})_S$ is an isomorphism between V and R^n.

Theorem 8.3.X A linear transformation $T : V \to W$ is an isomorphism if and only if whenever $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for V, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is a basis for W.

Examples of Natural isomorphisms

Between P_{n-1} and R^n: $a_0 + a_1x + \dots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \dots, a_{n-1})$

Between M_{22} and R^4: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a,\ b,\ c,\ d)$